# Note on the Cohomology of Color Hopf and Lie Algebras

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**Abstract**

Let $A$ be a $(G, \chi)$-Hopf algebra with bijective antipode and let $M$ be a $G$-graded $A$-bimodule. We prove that there exists an isomorphism

$$\mathrm{HH}^*_{\mathrm{gr}}(A, M) \cong \mathrm{Ext}^*_{A\text{-}\mathrm{gr}}(\mathbb{K}, {}^{ad}(M)),$$

where $\mathbb{K}$ is viewed as the trivial graded $A$-module via the counit of $A$, ${}^{ad}M$ is the adjoint $A$-module associated to the graded $A$-bimodule $M$ and $\mathrm{HH}^*_{\mathrm{gr}}$ denotes the $G$-graded Hochschild cohomology. As an application, we deduce that the graded cohomology of color Lie algebra $L$ is isomorphic to the graded Hochschild cohomology of its universal enveloping algebra $U(L)$, solving a question of M. Scheunert.

## 1 Introduction

Color Lie algebras have been introduced in [9] and studied systematically in [10, 11, 12, 13]. Some recent interest relates to their representation theory and related graded ring theory, [3]. The Cartan-Eilenberg cohomology theory for Lie algebras [1, 2], has been extended to color Lie algebras by Scheunert and Zhang in [12, 13]. In this note we introduce a graded cohomology of Color Lie algebras which coincide, in the case of degree zero, with the graded cohomology of $L$

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defined by Scheunert and Zhang. We show an isomorphism between the graded Hochschild cohomology of the universal enveloping algebra and the Lie cohomology for arbitrary color Lie algebras, which amounts to a careful manipulation of the group grading structure involved. We start from an abelian group $G$ with a skew-symmetric bicharacter $\varepsilon$. Consider a $G$-graded $\varepsilon$-Lie algebra $L$. Denote by $U(L)$ its universal enveloping algebra. Note that $U(L)$ has a natural $(G, \varepsilon)$-Hopf algebra structure in the sense of Definition 1. Therefore, first, we study the cohomology of arbitrary $(G, \chi)$-Hopf algebras, where $\chi$ is any bicharacter of $G$. We prove the main result concerning the (graded) Hochschild cohomology of a $(G, \chi)$-Hopf algebra with bijective antipode (see Theorem 2). Then, we combine this with the information coming from the color Kozsul resolution of the trivial module $\mathbb{K}$ of color Lie algebra $L$ to get the desired result for color Lie algebras (see Theorem 4). Our theorem extends the result of Cartan-Eilenberg for Lie algebras (see [2], pp. 277) and solves a question of M. Scheunert in the case (of degree zero) ([12]).

This note is organized as follows: in Section 2 we fix notation and provide background material concerning finite group gradings and color Lie algebras; in Section 3 we study $(G, \chi)$-Hopf algebras in detail and prove the main theorem Theorem 2; in Section 4 we study the color Koszul resolution of the trivial module $\mathbb{K}$ of color Lie algebra $L$ (Theorem 3), and, by using it, we obtain Theorem 4.

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# 2 Premilinaries

Throughout this paper groups are assumed to be abelian and $\mathbb{K}$ is a field of characteristic zero. We recall some notation for graded algebras and graded modules [8], and some facts on color Lie algebras from [10, 11, 12, 13].

## 2.1 Graded Hochschild cohomology

Let $G$ be an abelian group with identity element $e$. We will write $G$ as an multiplicative group.

An associative algebra $A$ with unit $1_A$, is said to be $G$-graded, if there is a family $\{A_g | g \in G\}$ of subspaces of $A$ such that $A = \oplus_{g\in G} A_g$ with $1_A \in A_e$ and $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$. Any element $a \in A_g$ is called homogeneous of degree $g$, and we write $|a| = g$.

A (left) graded $A$-module $M$ is a left $A$-module with an decomposition $M = \oplus_{g\in G} M_g$ such that $A_g.M_h \subseteq M_{gh}$. Let $M$ and $N$ be graded $A$-modules. Define

$$\mathrm{Hom}_{A\text{-gr}}(M, N) = \{f \in \mathrm{Hom}_A(M, N) |\ \ f(M_g) \subseteq N_g, \quad \forall\ \ g \in G\}\,. \tag{2.1}$$

We obtain the category of graded left $A$-modules, denoted by $A$-gr (see [8]). Denote by $\mathrm{Ext}^n_{A\text{-gr}}(-,-)$ the $n$-th right derived functor of the functor $\mathrm{Hom}_{A\text{-gr}}(-,-)$.

Let us recall the notion of graded Hochschild cohomology of a graded algebra $A$. A graded $A$-bimodule is a $A$-bimodule $M = \oplus_{g\in G} M_g$ such that $A_g.M_h.A_k \subseteq M_{ghk}$. Similar as the above, we obtain the category of graded $A$-bimodules, denoted by $A$-$A$-gr.

Let $A^e = A \otimes_{\mathbb{K}} A^{op}$ be the enveloping algebra of $A$, where $A^{op}$ is the opposite algebra of $A$. Note that the algebra $A^e$ also is graded by $G$ by setting $A^e_g := \sum_{h\in G} A_h \otimes_{\mathbb{K}} A_{h^{-1}g}$.

Now the graded $A$-bimodule $M$ becomes a graded left $A^e$-module just by defining the $A^e$-action as

$$(a \otimes a')m = a.m.a', \tag{2.2}$$

and it is clear that $A^e_g M_h \subseteq M_{gh}$, i.e., $M$ is a graded $A^e$-module. Moreover, every graded left $A^e$-module arises in this way. Precisely, the above correspondence establishes an equivalence of categories

$$A\text{-}A\text{-gr} \simeq A^e\text{-gr}. \tag{2.3}$$

In the sequel we will identify these categories.

Let $M$ be a graded $A$-bimodule, equivalently, graded left $A^e$-module. The $n$-th graded Hochschild cohomology of $A$ with value in $M$ is defined by

$$\mathrm{HH}^n_{\mathrm{gr}}(A, M) := \mathrm{Ext}^n_{A^e\text{-gr}}(A, M), \quad n \geq 0, \tag{2.4}$$

where $A$ is the graded left $A^e$-module induced by the multiplication of $A$, and the algebra $A^e = \oplus_{g\in G} A^e_g$ is considered as a $G$-graded algebra as above.

## 2.2 Color Lie algebras

The concept of color Lie algebras is related to an abelian group $G$ and an anti-symmetric bicharacter $\varepsilon : G \times G \to \mathbb{K}^\times$, i.e.,

$$\varepsilon(g, h)\,\varepsilon(h, g) = 1, \tag{2.5}$$

$$\varepsilon(g, hk) = \varepsilon(g, h)\,\varepsilon(g, k), \tag{2.6}$$

$$\varepsilon(gh, k) = \varepsilon(g, k)\,\varepsilon(h, k), \tag{2.7}$$

where $g, h, k \in G$ and $\mathbb{K}^\times$ is the multiplicative group of the units in $\mathbb{K}$.

A $G$-graded space $L = \oplus_{g\in G} L_g$ is said to be a $G$-graded $\varepsilon$-Lie algebra (or simply, color Lie algebra), if it is endowed with a bilinear bracket $[-,-]$ satisfying the following conditions

$$[L_g, L_h] \subseteq L_{gh}, \tag{2.8}$$

$$[a, b] = -\varepsilon(|a|, |b|)\,[b, a], \tag{2.9}$$

$$\varepsilon(|c|, |a|)\,[a, [b, c]] + \varepsilon(|a|, |b|)\,[b, [c, a]] + \varepsilon(|b|, |c|)\,[c, [a, b]] = 0, \tag{2.10}$$

where $g, h \in G$, and $a, b, c \in L$ are homogeneous elements.

For example, a super Lie algebra is exactly a $\mathbb{Z}_2$-graded $\varepsilon$-Lie algebra where

$$\varepsilon(i,j) = (-1)^{ij}, \quad \forall \quad i,j \in \mathbb{Z}_2. \tag{2.11}$$

Let $L$ be a color Lie algebra as above and $T(L)$ the tensor algebra of the underlying $G$-graded vector space $L$. It is well-known that $T(L)$ has a natural $\mathbb{Z}\times G$-grading which is fixed by the condition that the degree of a tensor $a_1 \otimes ... \otimes a_n$ with $a_i \in L_{g_i}$, $g_i \in G$, $1 \leq i \leq n$, is equal to $(n, g_1 \cdots g_n)$. The subspace of $T(L)$ spanned by homogeneous tensors of order $\leq n$ will be denoted by $T^n(L)$. Let $J(L)$ be the $G$-graded two-sided ideal of $T(L)$ which is generated by

$$a \otimes b - \varepsilon(|a|,|b|)\, b \otimes a - [a,b] \tag{2.12}$$

with homogeneous $a, b \in L$. The quotient algebra $U(L) := T(L)/J(L)$ is called the universal enveloping algebra of the color Lie algebra $L$. The $\mathbb{K}$-algebra $U(L)$ is a $G$-graded algebra and has a positive filtration by putting $U^n(L)$ equal to the canonical image of $T^n(L)$ in $U(L)$.

In particular, if $L$ is $\varepsilon$-commutative (i.e., $[L,L] = 0$), then $U(L) = S(L)$ (the $\varepsilon$-symmetric algebra of the graded space $L$).

The canonical map $\mathtt{i} : L \to U(L)$ is a $G$-graded homomorphism and satisfies

$$\mathtt{i}(a)\,\mathtt{i}(b) - \varepsilon(|a|,|b|)\,\mathtt{i}(b)\,\mathtt{i}(a) = \mathtt{i}([a,b]). \tag{2.13}$$

The $\mathbb{Z}$-graded algebra $G(L)$ associated with the filtered algebra $U(L)$ is defined by letting $G^n(L)$ be the vector space $U^n(L)/U^{n-1}(L)$ and $G(L)$ the space $\oplus_{n\in\mathbb{N}} G^n(L)$ (note $U^{-1}(L) := \{0\}$). Consequently, $G(L)$ is a $\mathbb{Z}\times G$-graded algebra. The well-known generalized Poincaré-Birkhoff-Witt theorem, [10], states that the canonical homomorphism $\mathtt{i} : L \to U(L)$ is an injective $G$-graded homomorphism; moreover, if $\{x_i\}_I$ is a homogeneous basis of $L$, where the index set $I$ well-ordered. Set $y_{k_j} := \mathtt{i}(x_{k_j})$, then the set of ordered monomials $y_{k_1} \cdots y_{k_n}$ is a basis of $U(L)$, where $k_j \leq k_{j+1}$ and $k_j < k_{j+1}$ if $\varepsilon(g_j, g_j) \neq 1$ with $x_{k_j} \in L_{g_j}$ for all $1 \leq j \leq n, n \in \mathbb{N}$. In case $L$ is finite-dimensional $U(L)$ is a two-sided (graded) Noetherian algebra (e.g., see [3]).

# 3 $(G,\chi)$-Hopf algebras, graded Hochschild cohomology

Through this section $G$ is an abelian group with a bicharecter $\chi : G\times G \longrightarrow \mathbb{K}^{\times}$. All unspecified graded spaces (algebras, coalgebras, ...) are graded by $G$; all unadorned Hom and tensor are taken over $\mathbb{K}$.

## 3.1 Twisted algebras

Let $(A = \oplus_{g\in G} A_g, \cdot, 1_A)$ be a graded algebra and $\chi$ be a bicharacter. Then there exists a new (graded) associative multiplication $\cdot^{\chi}$ on the $\mathbb{K}$-space $\oplus_{g\in G} A_g$ defined by

$$a \cdot^{\chi} b = \chi(|a|,|b|) a \cdot b \tag{3.1}$$

with $a, b$ homogeneous elements. It is easy to see that $(\oplus_{g\in G} A_g, \cdot_{\chi}, 1_A)$ is a (graded) associative algebra, which will be called the twisted algebra of $A$ by the bicharacter $\chi$ and will be denoted by $A^{\chi}$.

Let $A = (\oplus_g A_g, \cdot, 1_A)$ be a graded algebra and $\chi$ be a bicharacter. Let $A^\chi$ be the twisted algebra of $A$ by $\chi$. Consider the opposite algebra $A^{op}$, and denote its multiplication by $\circ$. Thus we may consider the algebra $(A^{op})^\chi$, the multiplication of which will be denoted by $\cdot_\chi$. Hence we have

$$a \cdot_\chi b = \chi(|a|,|b|) a \circ b = \chi(|a|,|b|) b \cdot a. \tag{3.2}$$

Let $A = (\oplus_{g\in G} A_g, \cdot, 1_A)$ be a graded algebra, and $\chi$ a bicharacter and $A^\chi = (\oplus_{g\in G} A_g, \cdot^\chi, 1_A)$ the corresponding twisted algebra of $A$. Let $M = \oplus_{g\in G} M_g$ be a graded $A$-module. Then there exists a new graded $A^\chi$-module structure, denoted by $.^\chi$, on the graded $\oplus_{g\in G} M_g$ defined by

$$a.^\chi m := \chi(|a|,|m|) a.m, \tag{3.3}$$

where $a \in A$ and $m \in M$ are homogeneous. Thus $\oplus_{g\in G} M_g$ becomes a right graded $A^\chi$-module, which will be denoted by $M^\chi$. Clearly every graded $A^\chi$-module arises in the way.

Therefore we have

**Proposition 1** *Use the above notation. There exists an equivalence of categories between* $A$-gr *and* $A^\chi$-gr.

For further use, we need to introduce: let $A$ and $B$ be graded algebras (by $G$), define a (graded) associative algebra structure $(A\otimes B)^\chi$ on the space $A \otimes B$, with the multiplication "$*$" given by the Lusztig's rule [6],

$$(a \otimes b) * (a' \otimes b') = \chi(|b|,|a'|) aa' \otimes bb', \tag{3.4}$$

where $a, a' \in A$ and $b, b' \in B$ are homogeneous.

### 3.2 Twisted coalgebras

Recall from [14] that a graded coalgebra $C$ is a graded space $C = \oplus_{g\in G} C_g$ with comultiplication $\Delta : C \to C \otimes C$, and counit $\epsilon : C \to \mathbb{K}$ satisfying the following conditions: $\Delta(C_g) \subseteq \sum_{h\in G} C_h \otimes C_{h^{-1}g}$, and $\epsilon(C_g) = 0$ for $g \neq e$, $g \in G$.

We define twisted coalgebras as follows: let $C = (C, \Delta, \epsilon)$ be a graded coalgebra, consider a new (graded) comultiplication $\Delta^\chi$ on $C$ defined by

$${}^\chi\Delta(c) = \sum \chi(|c_1|,|c_2|)(c_1 \otimes c_2) \tag{3.5}$$

where $\Delta(c) = \sum_c c_1 \otimes c_2$ is Sweedler's notation with all factors $c_1$, $c_2$ homogeneous. It is easy to check that $(C, {}^\chi\Delta, \epsilon)$ is a (graded) colagebra, it will be denoted by ${}^\chi C$.

Note that the opposite coalgebra of ${}^\chi C$, denoted by $({}^\chi C)^{cop}$ will have the comultiplication as follows

$${}_\chi\Delta(c) = \sum \chi(|c_1|,|c_2|)(c_2 \otimes c_1). \tag{3.6}$$

Let $C$ be a graded coalgebra. Denote by gr-$C$ the category of graded right $C$-comodules, with morphisms being graded homomorphism of comodules (of degree $e$).

Dually to Proposition 1, we obtain

**Proposition 2** *Use the above notation. There exists an equivalence of categories between* gr-$C$ *and* gr-${}^{\chi}C$*.*

We need the following construction: let $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ be two graded coalgebras, then the following law

$$ {}^{\chi}(\Delta_C \otimes \Delta_D)(c \otimes d) = \sum \chi(|c_2|, |d_1|)\,(c_1 \otimes d_1) \otimes (c_2 \otimes d_2) \tag{3.7} $$

defines on the space $C \otimes D$ a structure of graded coalgebra with counit $\epsilon_C \otimes \epsilon_D$, which is denoted by ${}^{\chi}(C \otimes D)$.

## 3.3 $(G, \chi)$-Hopf-algebras

**Definition 1** *A $(G, \chi)$-Hopf algebra $A$ (compare [5] and [7], p.206) is a 5-tuple $(A, m, \eta, \Delta, \epsilon, S)$ such that*
*(**T**1): $A = \oplus_{g \in G} A_g$ is a graded algebra with multiplication $m : A \otimes A \longrightarrow A$ and the unit map $\eta : K \longrightarrow A$. In the meantime, $(A, \Delta, \epsilon)$ is a graded coalgebra with respect to the same grading.*
*(**T**2): The counit $\epsilon : A \longrightarrow K$ is an algebra map. The comultiplication $\Delta : A \longrightarrow (A \otimes A)^{\chi}$ is an algebra map, where the algebra $(A \otimes A)^{\chi}$ is defined as in (3.4).*
*(**T**3): The antipode $S : A \longrightarrow A$ is a graded map such that*

$$ \sum a_1 S(a_2) = \epsilon(a) = \sum S(a_1) a_2 \tag{3.8} $$

*for all homogeneous $a \in A$, where we use Sweedler's notation $\Delta(a) = \sum a_1 \otimes a_2$.*

**Remark 1**

1. *A 4-tuple $(A, m, \eta, \Delta, \epsilon)$ satisfying the (**T**1) and (**T**2) will be called a $(G, \chi)$-bialgebra.*
2. *The condition (**T**2) implies exactly that the following holds:*

$$ \epsilon(1_A) = 1, \quad \epsilon(aa') = \epsilon(a)\epsilon(a'), \tag{3.9} $$
$$ \Delta(1_A) = 1_A \otimes 1_A, \tag{3.10} $$
$$ \Delta(aa') = \sum \chi(|a_2|, |a'_1|)\, a_1 a'_1 \otimes a_2 a'_2 = \Delta(a) * \Delta(b), \tag{3.11} $$

   *where $1_A$ is the identity element of $A$, and $a, a' \in A$ are homogeneous. Note that these four equations exactly state that*

$$ \eta : \mathbb{K} \longrightarrow A, \quad m : {}^{\chi}(A \otimes A) \longrightarrow A \tag{3.12} $$

   *are coalgebra maps, where the coalgeba ${}^{\chi}(A \otimes A)$ is defined in (3.6).*

3. *A Hopf ideal of a $(G, \chi)$-Hopf algebra $A$ is a graded ideal $I \subseteq A$ and a coideal (i.e., $\Delta(I) \subseteq A \otimes I + I \otimes A$ and $\epsilon(I) = 0$) satisfying $S(I) \subseteq I$.*

   *Thus there exist a unique $(G, \chi)$-Hopf algebra structure on the space $A/I$ such that the canonical map $\pi : A \longrightarrow A/I$ is a $(G, \chi)$-Hopf algebra morphism.*

Let $(A, m, \eta)$ and $(C, \Delta, \epsilon)$ be a graded algebra and a graded coalgebra respectively. Then $\mathrm{Hom}(C, A)$ becomes an associative algebra with the convolution product $\star$ defined by

$$(f \star g)(c) = m \circ (f \otimes g) \circ \Delta(c) = \sum f(c_1)g(c_2), \tag{3.13}$$

for $f, g \in \mathrm{Hom}(C, A)$, $c \in C$. Note that the unit of $\mathrm{Hom}(C, A)$ is $\eta \circ \epsilon$. Moreover, it is easy to see that $\mathrm{Hom}_{\mathrm{gr}}(C, A)$ is a subalgebra of $\mathrm{Hom}(C, A)$, i.e., if $f$ and $g$ are graded maps (of degree $e$), then so is $f \star g$.

Let $A = (A, m, \eta, \Delta, \epsilon, S)$ be a $(G, \chi)$-Hopf algebra. Consider the algebra $\mathrm{Hom}(A, A)$ with the convolution product $\star$. Then the condition (**T**3) is equivalent to

$$S \star \mathrm{Id_A} = \eta \circ \epsilon = \mathrm{Id_A} \star S. \tag{3.14}$$

This shows the uniqueness of the antipode $S$. Now we obtain a result similar to the one in [14], p.74 (compare [5], Theorem 2.10).

**Lemma 1**

1. *The antipode* $S : (A, \cdot, 1_A) \longrightarrow ((A^{op})^{\chi}, \cdot_{\chi}, 1_A)$ *is an algebra morphism with* $\cdot_{\chi}$ *defined by equation (3.2).*
2. *The antipode* $S : ({}^{\chi}A)^{cop} \longrightarrow A$ *is a coalgebra morphism, where the comultiplication of* $({}^{\chi}A)^{cop}$ *is defined by (3.6).*

**Proof:** We will imitate the proof in [14], and we will only prove the first statement, since the second can be proved similarly.

One see $S(1_A) = 1_A$ by the condition (**T**3). Now it suffices to show that $S(aa') = S(a) \cdot_{\chi} S(a') = \chi(|a|, |a'|)S(a')S(a)$ for all homogeneous elements $a, a' \in A$. Consider the convolution algebra $\mathrm{Hom}({}^{\chi}(A \otimes A), A)$, where the coalgebra structure of ${}^{\chi}(A \otimes A)$ is defined as in (3.7). Now define two elements $F, G \in \mathrm{Hom}({}^{\chi}(A \otimes A), A)$ by

$$F(a \otimes a') = S(aa') \quad \text{and} \quad G(a \otimes a') = \chi(|a|, |a'|)S(a')S(a). \tag{3.15}$$

So we need to show that $F = G$. We claim that

$$F \star m = m \star G = \eta \circ (\epsilon \otimes \epsilon) \tag{3.16}$$

in the convolution algebra $\mathrm{Hom}({}^{\chi}(A \otimes A), A)$, where $m$ denotes the multiplication of $A$. Indeed that $\eta \circ (\epsilon \otimes \epsilon)$ is the unit in the convolution algebra $\mathrm{Hom}({}^{\chi}(A \otimes A), A)$, thus we obtain $F = G$, as required.

In fact, by the definition of the convolution product $\star$ and the coalgebra structure of ${}^{\chi}(A \otimes A)$, we have

$$
\begin{aligned}
(F \star m)(a \otimes a') &= m \circ (F \otimes m) \circ {}^{\chi}(\Delta \otimes \Delta)(a \otimes a') && (3.17)\\
&= \sum \chi(|a_2|, |a'_1|)\, S(a_1 a'_1) a_2 a'_2 \quad \text{By Remark 1} && (3.18)\\
&= m(S \otimes \mathrm{Id})\Delta(aa') && (3.19)\\
&= \epsilon(ab) = (\eta \circ (\epsilon \otimes \epsilon))(a \otimes a'). && (3.20)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(m\star G)(a\otimes a') &= m\circ(m\otimes G)\circ {}^{\chi}(\Delta\otimes\Delta)(a\otimes a') && (3.21)\\
&= \sum \chi(|a_2|,|a_1'|)\, a_1a_1'G(a_2\otimes a_2') && (3.22)\\
&= \sum \chi(|a_2|,|a_1'|)\chi(|a_2|,|a_2'|)\, a_1a_1'S(a_2')S(a_2) \quad \text{By } |a'|=|a_1'||a_2'| && (3.23)\\
&= \sum \chi(|a_2|,|a'|)\, a_1a_1'S(a_2')S(a_2) && (3.24)\\
&= \sum \chi(|a_2|,|a'|) a_1S(a_2)\epsilon(a') && (3.25)\\
&= \epsilon(a)\epsilon(a') = (\eta\circ(\epsilon\otimes\epsilon))(a\otimes a'). && (3.26)
\end{aligned}
$$

(Here the second to the last equality follows from the fact that if $|a'|\neq e$ then $\epsilon(a')=0$; otherwise, $\chi(|a_2|,|a'|)=1$.) Thus we arrive at

$$(F\star m)=\eta\circ(\epsilon\otimes\epsilon)=(m\star G). \tag{3.27}$$

This completes the proof. □

**Remark 2** *If the antipode $S$ is bijective with inverse $S^{-1}$, then by Lemma 1, we have:*

$$S^{-1}(a)S^{-1}(a')=\chi(|a|,|a'|)S^{-1}(a'a). \tag{3.28}$$

*In this case, we call such a $(G,\chi)$-Hopf algebra a color Hopf algebra.*

The following result is quite useful when we construct an antipode on a $(G,\chi)$-bialgebra.

**Lemma 2** *Let $(A,m,\eta,\Delta,\epsilon)$ be a $(G,\chi)$-bialgebra generated by a set $\Lambda$ of homogeneous elements (as an algebra). If there exists an algebra morphism $S: A\longrightarrow (A^{op})^{\chi}$ such that each (3.8) holds for each $a\in\Lambda$, then $S$ is the antipode of $A$.*

**Proof:** We just need to check the (3.8) for all elements in $A$. For this, it suffices to show that if two homogeneous elements $a,b\in A$ satisfy the (3.8), so does $ab$.

In fact, by Remark 1 and then Lemma 1

$$
\begin{aligned}
\sum(ab)_1S((ab)_2) &= \sum \chi(|a_2|,|b_1|)a_1b_1S(a_2b_2) && (3.29)\\
&= \sum \chi(|a_2|,|b_1|)\chi(|a_2|,|b_2|)a_1b_1S(b_2)S(a_2) && (3.30)\\
&= \sum \chi(|a_2|,|b|)\epsilon(a)\epsilon(b) && (3.31)\\
&= \epsilon(a)\epsilon(b)=\epsilon(ab). && (3.32)
\end{aligned}
$$

(The third equality uses the fact that $|b|=|b_1|\cdot|b_2|$; the fourth equality uses the fact that $\epsilon(b)\neq 0$ implies $\chi(|a_2|,|b|)=1$.) In a similar way we may establish the right hand side of the (3.8). This completes the proof. □

**Example**

Using Lemma 2, we will give an important example of color Hopf algebras. Let $V$ be a $G$-graded space and denote by $T(V)$ the tensor algebra on $V$. Thus $T(V)$ is a $G$-graded algebra generated by $V$. By the universal property of $T(V)$, there exist unique graded algebra morphisms

$$\Delta : TV \longrightarrow (T(V) \otimes T(V))^{\chi} \quad v \longmapsto 1 \otimes v + v \otimes 1, \tag{3.33}$$

$$\epsilon : T(V) \longrightarrow \mathbb{K} \quad v \longmapsto 0, \tag{3.34}$$

for all $v \in V$.

To show that $(T(V), \Delta, \epsilon)$ is a (graded) coalgebra, we need to verify

$$(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta \quad \text{and} \quad (\epsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id} = (\mathrm{Id} \otimes \epsilon) \circ \Delta \tag{3.35}$$

where Id the identity map of $T(V)$.

Note that all above maps are algebra morphisms, so it suffices to check them on a set of generators of the algebra $T(V)$. Clearly all elements in $V$ satisfy the above equations, thus we have shown that $(T(V), \Delta, \epsilon)$ is an coalgebra, hence $TV$ is a $(G, \chi)$-bialgebra.

Again by the universal property of $T(V)$, there exists a unique algebra map

$$S : T(V) \longrightarrow (T(V)^{op})^{\chi} \quad v \longmapsto -v, \tag{3.36}$$

for all $v \in V$. Now applying Lemma 2, we deduce that $T(V)$ is a color Hopf algebra. We call the resulting color Hopf algebra $T(V)$ the tensor color Hopf algebra of $V$.

It follows from Remark 1 that if $I$ is a Hopf ideal of $T(V)$, then we have a quotient $(G, \chi)$-Hopf algebra $T(V)/I$.

An important example is as follows: let $L$ be a $G$-graded $\varepsilon$-Lie algebra, then its universal enveloping algebra $U(L) = T(L)/J(L)$ is $(G, \varepsilon)$-Hopf algebra with $J(L)$ defined by (2.12), since $J(L)$ is a Hopf idea. Explicitly, $U(L)$ is a color Hopf algebra with comultiplication $\Delta$ and $\epsilon$ given by

$$\Delta(a) = 1 \otimes a + a \otimes 1, \quad \epsilon(a) = 0, \quad \forall \quad a \in L. \tag{3.37}$$

We now consider graded right $A$-modules. As before $A$ will be a $(G, \chi)$-Hopf algebra. Recall that a right gr-free $A$-module of the form $V \otimes A$, where $V$ is a graded space and $V \otimes A$ is graded by assigning to $v \otimes a$ the degree $|v| \cdot |a|$, for all homogeneous elements $v \in V$ and $a \in A$, and the right action is given by (see [8])

$$(v \otimes a)a' = v \otimes aa'. \tag{3.38}$$

In fact, gr-free modules are just the free objects in the category of graded right $A$-modules.

Since $\Delta : A \longrightarrow (A \otimes A)^{\chi}$ is an algebra map, the algebra $(A \otimes A)^{\chi}$ becomes a graded right $A$-module. Explicitly, the right $A$-action on $(A \otimes A)^{\chi}$ is given by

$$(a \otimes a')b := (a \otimes a') * \Delta(b) = \sum \chi(|a'|, |b_1|) ab_1 \otimes a'b_2, \tag{3.39}$$

for homogeneous $a, a', b \in A$ and $\Delta(b) = \sum b_1 \otimes b_2$ is the Sweedler notation. Note that the grading of $(A \otimes A)^\chi$ is given such that the degree of $a \otimes a'$ is $|a| \cdot |a'|$.

The following result will be essential.

**Proposition 3** *The right $A$-module $(A \otimes A)^\chi$ defined above is gr-free.*

**Proof:** Let $V$ denote the underlying graded space of $A$. Thus $V \otimes A$ becomes a right gr-free module. Define a map $\Psi : (A \otimes A)^\chi \longrightarrow V \otimes A$ by

$$\Psi(a \otimes a') = \sum aS(a'_1) \otimes a'_2, \tag{3.40}$$

where $a, a' \in A$ are homogeneous and $\Delta(a) = \sum a'_1 \otimes a'_2$ is the Sweedler notation. It is obvious that $\Psi$ is a (graded) bijective map with inverse

$$\Psi^{-1}(a \otimes a') = \sum aa'_1 \otimes a'_2.$$

We claim that $\Psi$ is a right $A$-module morphism, then we are done.

In fact we have

$$\Psi((a \otimes a')b) = \sum \chi(|a'|, |b_1|)\Psi(ab_1 \otimes a'b_2) \tag{3.41}$$

$$= \sum \chi(|a'|, |b_1|) ab_1 S((a'b_2)_1) \otimes (a'b_2)_2. \tag{3.42}$$

By Remark 1 we have

$$\Psi((a \otimes a')b) = \sum \chi(|a'|, |b_1|)\chi(|a'_2|, |b_2|) ab_1 S(a'_1 b_2) \otimes a'_2 b_3, \tag{3.43}$$

where we use $(\Delta \otimes \mathrm{Id}_A)\Delta(b) = \sum b_1 \otimes b_2 \otimes b_3$ (see [14]).

By Lemma 1, we have $S(a'_1 b_2) = \chi(|a'_1|, |b_2|) S(b_2) S(a'_1)$, hence

$$\Psi((a \otimes a')b) = \sum \chi(|a'|, |b_1|)\chi(|a'_2|, |b_2|)\chi(a'_1, b_2) ab_1 S(b_2) S(a'_1) \otimes a'_2 b_3 \tag{3.44}$$

$$= \sum \chi(|a'|, |b_1||b_2|) ab_1 S(b_2) S(a'_1) \otimes a'_2 b_3 \tag{3.45}$$

$$= \sum \chi(|a'|, |b_1|) a\epsilon(b_1) S(a_1) \otimes a'_2 b_2. \tag{3.46}$$

Using the fact that $\epsilon(b_1) \neq 0$ implies that $|b_1| = e$ and hence $\chi(|a'|, |b_1|) = 1$, we obtain

$$\sum \chi(|a'|, |b_1|) a\epsilon(b_1) S(a_1) \otimes a'_2 b_2 = \sum aS(a'_1) \otimes a'_2 b_2 \tag{3.47}$$

$$= \sum aS(a'_1) \otimes a'_2 b. \tag{3.48}$$

Note the right $A$-module structure on $V \otimes A$. So we have proved that

$$\Psi((a \otimes a')b) = (\Psi(a \otimes a'))b, \quad \forall \quad a, a', b \in A. \tag{3.49}$$

This completes the proof. $\square$

We obtain

**Theorem 1** *Let $(A, m, \eta, \Delta, \epsilon, S)$ be a color Hopf algebra. Then the categories $A$-$A$-*gr *and $(A \otimes A)^\chi$-*gr *are equivalent.*

**Proof:** We are going to construct the functor

$$F : A\text{-}A\text{-gr} \rightsquigarrow (A \otimes A)^{\chi}\text{-gr} \tag{3.50}$$

as follows: let $M$ be a graded $A$-bimodule, we denote the two-sided $A$-action on $M$ by ".". Define $F(M) = M$ as graded spaces with the left $(A \otimes A)^{\chi}$-action given by

$$(a \otimes a')m = \chi(|a'|, |m|)a.m.S(a'), \tag{3.51}$$

where $a, a' \in A$ and $m \in M$ are homogeneous. We claim that the action is well-defined, i.e.,

$$((a \otimes a') * (b \otimes b'))m = (a \otimes a')((b \otimes b')m), \tag{3.52}$$

where $*$ denotes the multiplication of the algebra $(A \otimes A)^{\chi}$ (see (3.4)).

In fact, we have

$$\begin{aligned} ((a \otimes a') * (b \otimes b'))m &= \chi(|a'|, |b|)(ab \otimes a'b')m &(3.53)\\ &= \chi(|a'|, |b|)\chi(|a'b'|, |m|)ab.m.S(a'b') &(3.54)\\ &= \chi(|a'|, |b|)\chi(|a'b'|, |m|)\chi(|a'|, |b'|)ab.m.S(b')S(a'). &(3.55)\end{aligned}$$

The last equality uses Lemma 1. 1. On the other hand,

$$\begin{aligned} (a \otimes a')((b \otimes b')m) &= \chi(|b'|, |m|)(a \otimes a')b.m.S(b') &(3.56)\\ &= \chi(|b'|, |m|)\chi(|a'|, |b.m.S(b')|)a.(b.m.S(b')).S(a'). &(3.57)\end{aligned}$$

Note that the degree of the element $b.m.S(b')$ is $|b| \cdot |m| \cdot |b'|$. By comparing the above two identities, we have proved the claim.

Conversely, we have the functor

$$G : (A \otimes A)^{\chi}\text{-gr} \rightsquigarrow A\text{-}A\text{-gr} \tag{3.58}$$

given as follows: let $N$ be a left graded $(A \otimes A)^{\chi}$-module, define $G(N)$ to be $N$ as graded spaces, and its $A$-bimodule structure given by

$$a.n = (a \otimes 1)n \quad \text{and} \quad a.a' = \chi^{-1}(|a'|, |n|)(1 \otimes S^{-1}(a'))n, \tag{3.59}$$

for all homogeneous $a, a' \in A$ and $n \in N$. Clearly, $G(N)$ is a left $A$-module. Note that

$$\begin{aligned} (n.a).b &= \chi^{-1}(|b|, |a| \cdot |n|)(1 \otimes S^{-1}(b))(n.a) &(3.60)\\ &= \chi^{-1}(|b|, |a| \cdot |n|)\chi^{-1}(|a|, |n|)((1 \otimes S^{-1}(b)) \star (1 \otimes S^{-1}(a)))n &(3.61)\\ &= \chi^{-1}(|b|, |a| \cdot |n|)\chi^{-1}(|a|, |n|)\chi(|b|, |a|)(1 \otimes S^{-1}(ab))n &(3.62)\\ &= \chi^{-1}(|ab|, |n|)(1 \otimes S^{-1}(ab))n = n.(ab). &(3.63)\end{aligned}$$

The third equality uses the fact $S^{-1}(b)S^{-1}(a) = \chi(|b|, |a|)S^{-1}(ab)$, see Remark 2, hence $G(N)$ is also a right $A$-module. Note that $(a.n).a' = a.(n.a')$, therefore, $G(N)$ is a graded $A$-bimodule. It is easy to check that the functors $F$ and $G$ are inverse to each other. Thus we have proved the result. □

## 3.4 Twisted Tensor Modules

In this subsection, we include some remarks and notation concerning tensor modules in the category $A$-gr for a given $(G, \chi)$-bialgebra $A$.

For given graded $A$-modules $M = \oplus_{g\in G} M_g$ and $N = \oplus_{g\in G} N_g$, we define a graded $A$-module $(M \otimes^\chi N)$ as follows: as graded spaces $(M \otimes^\chi N)$ coincides with $M \otimes N$ (note that the degree of $m \otimes n$ is just $|m| \cdot |n|$ for homogeneous $m \in M$ and $n \in N$); the (left) $A$-action is given by

$$a(m \otimes n) := \sum \chi(|a_2|, |m|) a_1 m \otimes a_2 n, \tag{3.64}$$

for all homogeneous elements $a \in A, m \in M, n \in N$ and $\Delta(a) = \sum a_1 \otimes a_2$ is the Sweedler notation. It is easy to check that $(M \otimes^\chi N)$ is a graded $A$-module.

Since $\epsilon : A \longrightarrow \mathbb{K}$ is an algebra map, then $\mathbb{K}$ becomes a graded $A$-module, which will be referred to as the trivial module (note that $\mathbb{K}$ is trivially graded). Thus the above defined "$(\otimes)^\chi$" has the following properties:

$$((M \otimes^\chi N) \otimes^\chi L) \simeq (M \otimes^\chi (N \otimes^\chi L)),$$
$$(M \otimes^\chi \mathbb{K}) \simeq M \simeq (\mathbb{K} \otimes^\chi M),$$

for all $M, N, L \in A$-gr.

## 3.5 Graded Hochschild Cohomology

Throughout this subsection, assume that $(A, m, \eta, \delta, \epsilon, S)$ is a color Hopf algebra.

Denote by $A$-gr (resp. $(A \otimes A)^\chi$-gr) the category of (left) graded $A$-modules (rep. $(A \otimes A)^\chi$-modules) with graded morphisms (of degree $e$). Note that here $(A \otimes A)^\chi$ is considered as an $G$-graded algebra as above.

Since $\Delta : A \longrightarrow (A \otimes A)^\chi$ is an algebra map, there is a restriction functor

$$\mathrm{Res} : (A \otimes A)^\chi\text{-gr} \rightsquigarrow A\text{-gr}. \tag{3.65}$$

More precisely, if $M$ is a graded $(A \otimes A)^\chi$-module, then $\mathrm{Res}(M) = M$ as graded spaces, and its left $A$-action is given by

$$am = (\sum a_1 \otimes a_2)m, \quad a \in A, \quad m \in M. \tag{3.66}$$

The following is a direct consequence of Proposition 3.

**Proposition 4** *The functor* $\mathrm{Res} : (A \otimes A)^\chi$-gr $\rightsquigarrow A$-gr *is exact and it preserves injective objects.*

**Proof:** The exactness of Res is obvious. Let $I$ be an injective object in $(A \otimes A)^\chi$-gr. To show that $\mathrm{Res}(I)$ is an injective object in $A$-gr, take any monomorphism

$$i : N \longrightarrow N' \tag{3.67}$$

and a morphism

$$f : N \longrightarrow \mathrm{Res}(I) \tag{3.68}$$

in $A$-gr, we claim that there exists some morphism

$$f' : N' \longrightarrow \mathrm{Res}(I) \tag{3.69}$$

such that $f'i = f$. then we are done. In fact, by Proposition 3, $(A \otimes A)^\chi$ is a gr-free right $A$-module, hence it is flat $A$-module. So we have a monomorphism of left $(A \otimes A)^\chi$-modules

$$j = (A \otimes A)^\chi \otimes_A i : (A \otimes A)^\chi \otimes_A N \longrightarrow (A \otimes A)^\chi \otimes_A N'. \tag{3.70}$$

Note that we have a morphism of right $(A \otimes A)^\chi$-modules

$$g : (A \otimes A)^\chi \otimes_A N \longrightarrow I, \quad (a \otimes a') \otimes n \longmapsto (a \otimes a')f(n), \tag{3.71}$$

where $a, a' \in A$ and $n \in N$. Since $I$ is an injective object in $(A \otimes A)^\chi$-gr, there exist a morphism

$$g' : (A \otimes A)^\chi \otimes_A N' \longrightarrow I \quad \text{such} \quad \text{that} \quad g' \circ j = g. \tag{3.72}$$

Define

$$f' : N' \longrightarrow \mathrm{Res}(I), \quad n' \longmapsto g'((1_A \otimes 1_A) \otimes n'), \tag{3.73}$$

where $n' \in N$ and $1_A \in A$ is the unit. Now it is easy to check that $f'i = f$ and this proves that $\mathrm{res}(I)$ is an injective object in $A$-gr. $\square$

Define the adjoint functor to be

$$^{ad}(-) = \mathrm{Res} \circ F : A\text{-}A\text{-gr} \rightsquigarrow A\text{-gr}.$$

Explicitly, let $M$ be a graded $A$-bimodule, then $^{ad}(M) = M$ as graded spaces, and the left $A$-module structure is given by

$$am = \sum \chi(|a_{(2)}|, |m|) a_1.m.S(a_2), \tag{3.74}$$

for homogeneous $a \in A$ and $m \in M$. The resulting graded $A$-module $^{ad}(M)$ is called the adjoint module associated the graded $A$-bimodule $M$.

The main theorem in this section is as follows:

**Theorem 2** *Let $A = (A, m, \eta, \Delta, \epsilon, S)$ be a color Hopf algebra and let $M$ be a graded $A$-bimodule. Then there exists an isomorphism*

$$\mathrm{HH}^n_{\mathrm{gr}}(A, M) \simeq \mathrm{Ext}^n_{A\text{-gr}}(\mathbb{K}, {}^{ad}(M)), \quad n \geq 0,$$

*where $\mathbb{K}$ is viewed as the trivial graded $A$-modules via the counit $\epsilon$, and $^{ad}(M)$ is the adjoint $A$-module associated to the graded $A$-bimodule $M$.*

**Proof:** First we show that there exists a natural isomorphism

$$\mathrm{Hom}_{A^e\text{-gr}}(A, -) \simeq \mathrm{Hom}_{A\text{-gr}}(\mathbb{K}, {}^{ad}(-)),$$

both of which are functors form $A$-$A$-gr to the category of vector spaces.

In fact, for each graded $A$-bimodule $M = \oplus_{g \in G} M_g$,

$$\mathrm{Hom}_{A^e\text{-gr}}(A, M) = \{m \in M_e | a.m = m.a, \quad \text{for all } a \in A\},$$

and

$$\mathrm{Hom}_{A\text{-gr}}(\mathbb{K}, {}^{ad}(M)) = \{m \in M_e | am = \epsilon(a)m, \quad \text{for all } a \in A\}.$$

We deduce that the isomorphism in using the definition of $am$, see (3.74)).

In general, for $n \geq 1$, we have

$$\begin{aligned}\mathrm{HH}^n_{\mathrm{gr}}(A,-) &= \mathrm{Ext}^n_{A^e\text{-gr}}(A,-)\\ &= R^n\mathrm{Hom}_{A^e\text{-gr}}(A,-),\end{aligned}$$

where $R^n$ means taking the $n$-th right derived functors. Now apply the above observation, we get

$$\mathrm{HH}^n_{\mathrm{gr}}(A,-) = R^n(\mathrm{Hom}_{A\text{-gr}}(\mathbb{K},-) \circ {}^{ad}(-)).$$

By Theorem 1 and Proposition 4, we obtain the functor ${}^{ad}(-)$ is exact and preserves injective objectives. Hence Grothendieck's spectral sequence (e.g., see [2] or [HS], p. 299) gives us

$$R^n(\mathrm{Hom}_{A\text{-gr}}(\mathbb{K},-) \circ {}^{ad}(-)) = R^n(\mathrm{Hom}_{A\text{-gr}}(\mathbb{K},-)) \circ {}^{ad}(-).$$

Hence, we have

$$\begin{aligned}\mathrm{HH}^n_{\mathrm{gr}}(A,-) &= R^n\mathrm{Hom}_{A\text{-gr}}(\mathbb{K},-) \circ {}^{ad}(-)\\ &= \mathrm{Ext}^n_{A\text{-gr}}(\mathbb{K},-) \circ {}^{ad}(-)\\ &= \mathrm{Ext}^n_{A\text{-gr}}(\mathbb{K}, {}^{ad}(-)).\end{aligned}$$

This completes the proof. □

## 3.6 Shift functor

We end this section with some observations on shift functor. Let $A$ be a $G$-graded algebra. For each $h \in G$, we define a shift functor $[h]$ from $A$-gr to itself as follows: for each $M \in A$-gr, define a graded $A$-module $M[h]$ by setting $(M[h])_g = M_{hg}$ for each $g \in G$. Note that $M[e] = M$.

Set

$$\mathrm{HOM}_A(M,N)_h = \{f \in \mathrm{Hom}_A(M,N) |\ \ f(M_g) \subseteq N_{gh}, \forall\ \ g \in G\}. \tag{3.75}$$

So we have (see [8], pp. 25):

$$\mathrm{HOM}_A(M,N)_h = \mathrm{Hom}_{A\text{-gr}}(M, N[h]). \tag{3.76}$$

For $M, N \in A$-gr, set

$$\mathrm{HOM}_A(M,N) := \oplus_{g\in G}\mathrm{HOM}_A(M,N)_h. \tag{3.77}$$

Let $\mathrm{EXT}^n_A(-,-)$ (resp. $\mathrm{EXT}^n_A(-,-)_h$) be the $n$-th right derived functor of the functor $\mathrm{HOM}_A(-,-)$ (resp. $\mathrm{HOM}_A(-,-)_h$).

Clearly, we have

$$\mathrm{EXT}^n_A(M,N)_h = \mathrm{Ext}^n_{A\text{-gr}}(M, N[h]), \quad n \geq 0 \tag{3.78}$$

and, consequently,

$$\mathrm{EXT}^n_A(M,N) = \oplus_{h\in G}\, \mathrm{Ext}^n_{A\text{-gr}}(M, N[h]), \quad n \geq 0. \tag{3.79}$$

Let $M$ be a graded $A$-bimodule which is regarded as a left $A^e$-module. Set

$$\mathrm{HH}^n(A, M)_h := \mathrm{EXT}^n_{A^e}(A, M)_h = \mathrm{HH}^n_{\mathrm{gr}}(A, M[h]), \quad n \geq 0. \tag{3.80}$$

From Theorem 2, we get immediately

**Corollary 1** *Under the hypotheses of Theorem 2. Then*

$$\mathrm{HH}^n(A, M)_h \simeq \mathrm{Ext}^n_{A\text{-gr}}(\mathbb{K}, {}^{ad}(M[h])) = \mathrm{EXT}^n_A(\mathbb{K}, {}^{ad}(M))_h$$

*for each* $h \in G$ *and* $n \geq 0$.

**Remark 3** *In the above two corollaries, we use the notation* ${}^{ad}(M[h])$. *Let us remark that, in general,* ${}^{ad}(M[h])$ *and* $({}^{ad}(M))[h]$ *are not isomorphic in* $A$-gr.

# 4 Graded Cartan-Eilenberg Cohomology

In this section we will extend the construction in [1] and [2] to color Lie algebras.

## 4.1 Color Kozsul Resolution

Let $L$ be a $G$-graded $\varepsilon$-Lie algebra over $\mathbb{K}$, and let $V$ be the underlying graded space of $L$. Set

$$\wedge_\varepsilon V := T(V) / \langle u \otimes v + \varepsilon(u, v) v \otimes u \rangle \tag{4.1}$$

where $u, v$ are homogeneous in $V$. Clearly $\wedge_\varepsilon V$ is graded by the group $\mathbb{Z} \times G$, in other words,

$$\wedge_\varepsilon V = \bigoplus_{n \geq 0} \wedge^n_\varepsilon V, \tag{4.2}$$

and each $\wedge^n_\varepsilon V$ is graded by $G$.

Define $C_n := U(L) \otimes_{\mathbb{K}} \wedge^n_\varepsilon V$, which is graded by $G$ such that the degree of $u \otimes v$ is $|u| \cdot |v|$, for homogeneous $u \in U(L)$ and $v \in \wedge^n_\varepsilon V$, $n \in \mathbb{N}$. Endow $C_n$ with a left $U(L)$-module structure, which is induced by the multiplication of $U(L)$. Obviously, each $C_n$ is a graded $U(L)$-module (with respect to the group $G$) and it is gr-free (again in the sense of [8]).

Denote by $\langle x_1, ..., x_n \rangle$ the element $x_1 \wedge \cdots \wedge x_n$ of $\wedge^n_\varepsilon V$. Define, for every homogeneous $y \in L$, an $L$-module homomorphism $\theta(y) : C_n \to C_n$ by

$$\begin{aligned} \theta(y)(u \otimes \langle x_1, ..., x_n \rangle) &:= -\varepsilon(|y|, |u|) uy \otimes \langle x_1, ..., x_n \rangle \\ &\quad + \sum_{i=1}^{n} \varepsilon(|y|, |u| \cdot |x_1| \cdots |x_{i-1}|) u \otimes \langle x_1, ..., [y, x_i], ..., x_n \rangle \end{aligned}$$

We claim that

$$\theta(x)\theta(y) - \varepsilon(|x|, |y|)\theta(y)\theta(x) = \theta([x, y]) \tag{4.3}$$

for all homogeneous $x, y, x_1, .., x_n \in L$ and $u \in U(L)$. We check that $\theta$ verifies Eq.(4.3) and the fact that $\theta = (\theta_1 \otimes \theta_2)^{\chi}$ is the twisted tensor product of $L$-module maps (see Section 3.4):

$$\theta_1(x) : U(L) \to U(L), u \mapsto \theta_1(x)(u) := -\varepsilon(x, u)ux$$

and

$$\theta_2(x) : \wedge^n L \to \wedge^n L, \langle x_1, ..., x_n \rangle \mapsto \sum_i^n \varepsilon(x, x_i) \langle x_1, ..., [x, x_i], ..., x_n \rangle .$$

We define also, for every $y \in L$, a graded $L$-module homogeneous homomorphism of degree zero $\sigma(y) : C_n \to C_{n+1}$ by

$$\sigma(y)(u \otimes \langle x_1, ..., x_n \rangle) := \varepsilon(|y|, |u|)u \otimes \langle y, x_1, ..., x_n \rangle$$

for all homogeneous elements $x_1, .., x_n \in L$ and $u \in U(L)$. It is easy to check that

$$\sigma([x, y]) = \theta(x)\sigma(y) - \varepsilon(x, y)\sigma(y)\theta(x). \tag{4.4}$$

Next we define by induction $L$-module homorphisms of degree zero $d_n : C_n \to C_{n-1}$ by

$$\sigma(y)d_{n-1} + d_n\sigma(y) = -\theta(y) \tag{4.5}$$

for all homogeneous elements $y \in L$ and $u \in U(L)$. We set $d_0 := 0$. Since $u \otimes \langle x_1, ..., x_n \rangle = \varepsilon(u, x_1)\sigma(x_1)(u \otimes \langle x_2, ..., x_n \rangle)$, it follows from Eq. (4.5) that

$$\begin{aligned} d_n(u \otimes \langle x_1, ..., x_n \rangle) &= \varepsilon(u, x_1)d_n\sigma(x_1)(u \otimes \langle x_2, ..., x_n \rangle) \\ &= \varepsilon(u, x_1)(-\theta(x_1) - \sigma(x_1)d_{n-1})(u \otimes \langle x_2, ..., x_n \rangle). \end{aligned}$$

We deduce that the operator $d_n$ is explicitly given by

$$\begin{aligned} & d_n(u \otimes \langle x_1, \cdots, x_n \rangle) \\ &= \sum_{i=1}^n (-1)^{i+1} \varepsilon_i \; u x_i \otimes \langle x_1, \cdots, \hat{x_i}, \cdots, x_n \rangle \\ &\quad + \sum_{1 \le i < j \le n} (-1)^{i+j} \varepsilon_i \varepsilon_j \varepsilon(|x_j|, |x_i|) \; u \otimes \langle [x_i, x_j], x_1, \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, x_n \rangle , \end{aligned}$$

for all homogeneous elements $u \in U(L)$ and $x_i \in L$, with $\varepsilon_i = \prod_{h=1}^{i-1} \varepsilon(|x_h|, |x_i|)$ $i \ge 2$, $\varepsilon_1 = 1$ and the sign $\widehat{\ }$ indicates that the element below it must be omitted. We will show preceding by induction on $n \in \mathbb{N}$ that

$$\theta(y)d_n = d_n\theta(y) \tag{4.6}$$

It is obvious if $n = 0$. For $n \ge 1$, we have

$$\begin{aligned} & \theta(y)d_n - d_n\theta(y)(u \otimes \langle x_1, ..., x_n \rangle) \\ &= \epsilon(u, x_1) \left( \theta(y)d_n\sigma(x_1) - d_n\theta(y)\sigma(x_1) \right) (u \otimes \langle x_2, ..., x_n \rangle). \end{aligned}$$

Since $\varepsilon(u, x_1) \neq 0$ for all homogeneous elements $x_1, u$, it sufficient to show that

$$\theta(y)\delta_n\sigma(x) - d_n\theta(y)\sigma(x) = 0.$$

On the other hand
$\theta(y)d_n\sigma(x) - d_n\theta(y)\sigma(x)$
$= -\theta(y)\theta(x) - \theta(y)\sigma(x)d_{n-1} - \varepsilon(y,x)d_n\sigma(x)\theta(y) - d_n\sigma([y,x])$, (by Eq. (4.5) and Eq.(4.4))
$= -\theta(y)\theta(x) - \theta(y)\sigma(x)d_{n-1} + \varepsilon(y,x)(\theta(x)\theta(y) + \sigma(x)d_{n-1}\theta(y)) + \theta\,[y,x]$
$+\sigma\,[y,x]\,d_{n-1})$, (by (4.5))
$= \{-\theta(y)\theta(x) + \varepsilon(y,x)\theta(x)\theta(y) + \theta\,[y,x]\}$
$+\{-\theta(y)\sigma(x) + \varepsilon(y,x)\sigma(x)\theta(y) + \sigma\,[y,x]\}\,d_{n-1}$, (by the induction hypothesis)
$= 0$, (by Eq. (4.3), and Eq. (4.4)).

Finally, we show that

$$d_{n-1}d_n = 0. \tag{4.7}$$

It is obvious that $d_0d_1 = 0$. We reason by induction. We have, for $n \geq 2$ :

$$d_{n-1}d_n(u \otimes \langle x_1 ... x_n\rangle) = \varepsilon(u, x_1)d_{n-1}d_n\sigma(x_1)(u \otimes \langle x_2...x_n\rangle),$$

from Eq. (4.5) we obtain:

$$\begin{aligned}
\varepsilon(u, x_1)d_{n-1}d_n\sigma(x_1)(u \otimes \langle x_2...x_n\rangle) &= -\varepsilon(u, x_1)d_{n-1}(\theta(x_1) + \sigma(x_1)d_{n-1}) \\
&= \varepsilon(u, x_1)d_{n-1}d_n\sigma(x_1)(u \otimes \langle x_2...x_n\rangle) \\
&= -\varepsilon(u, x_1)d_{n-1}(\theta(x_1) + \sigma(x_1)d_{n-1}) \\
&= -\varepsilon(u, x_1)(d_{n-1}\theta(x_1) + d_{n-1}\theta(x_1) \\
&+ \sigma(x_1)d_{n-2}d_{n-1}) \\
&= 0
\end{aligned}$$

from Eq. (4.6) and the induction hypothesis.

Let $\{x_i\}_I$ be a homogeneous basis of $L$, where $I$ is a well-ordered set. By the generalized PBW theorem the elements

$$e_{k_1} \cdots e_{k_m} \otimes \langle e_{l_1} \cdots e_{l_n}\rangle \tag{4.8}$$

with

$$k_1 \leq \cdots \leq k_m \quad \text{and} \quad k_i < k_{i+1} \quad \text{if} \quad \varepsilon(|e_{k_i}|, |e_{k_i}|) = -1 \tag{4.9}$$

and

$$l_1 \leq \cdots \leq l_n \quad \text{and} \quad l_i < l_{i+1} \quad \text{if} \quad \varepsilon(|e_{l_i}|, |e_{l_i}|) = 1 \tag{4.10}$$

form a homogeneous basis of $C_n$.

We define a family of $G$-graded subspace $F_pC$ of $C$ with $p \in \mathbb{Z}$, as follows: $(F_pC)_{-1} := \mathbb{K}$ and $(F_pC)_n$, $n \geq 0$, is the subspace of $C_n$ generated by the homogeneous basis (4.8) with $m + n \leq p$. We see that for all $n \geq 0$ the differential $d_n$ maps $(F_pC)_n$ into $(F_pC)_{n-1}$, then $F_pC$ is a $G$-graded subcomplex of $C$. For every $p \geq 1$ we define a $G$-graded complex $W^p$ by $(F_pC)_n/(F_pC)_{n-1}$ for $n \geq 0$ and $W^p_{-1} := \mathbb{K}$. It is now clear that the differential $d^p_n : W^p_n \to W^p_{n-1}$ is $G$-graded and given by

$d^p_n(e_{k_1}\cdots e_{k_m}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle)\equiv$

$$\sum_{i=1}^{n}(-1)^{i+1}\prod_{h=1}^{i-1}\varepsilon(|e_{l_h}|,|e_{l_i}|)\; e_{k_1}\;\cdots e_{k_m}e_{l_i}\otimes\langle e_{l_1},\cdots,\hat{e_{l_i}},\cdots,e_{l_n}\rangle \bmod (F_{p-1}C)_{n-1}.$$

Note that the summands on the right hand side are not necessarily of the form (4.8), since we cannot guarantee $k_m\leq l_i$.

**Lemma 3** *We have that the homology groups* $\mathrm{H}_n(W^p)=0$ *for all* $p\geq 1$ *and all* $n$*.*

**Proof:** We define a $G$-graded homomorphisms $t^p_n$ as follows: $t^p_{-1}:\mathbb{K}\to W^p_0$ is given by $t^p_{-1}(1_{\mathbb{K}}):=1\otimes\langle\rangle$ and, for $n\geq 0$, we define $t^p_n:W^p_n\to W^p_{n+1}$ by

$$t^p_n(e_{k_1}\cdots e_{k_m}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle)$$

$$\equiv\sum_{i=1}^{m}\prod_{h=i+1}^{m}\varepsilon|e_{k_i}|,|e_{k_h}|)\; e_{k_1}\cdots\hat{e_{k_i}}\cdots e_{k_m}\otimes\langle e_{k_i},e_{l_1},\cdots,e_{l_n}\rangle \bmod (F_{p-1}C)_n.$$

We will show that

$$d^p_{n+1}t^p_n+t^p_{n-1}d^p_n=p\;\mathrm{id}.$$

$$\begin{aligned}
&d^p_{n+1}t^p_n(e_{k_1}\cdots e_{k_m}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle)\\
&=\sum_{i,j=1}^{m,n}(-1)^j\prod_{h=i+1}^{m}\varepsilon(|e_{k_i}|,|e_{k_h}|)\,\varepsilon(|e_{k_i}|,|e_{l_j}|)\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\\
&\qquad e_{k_1}\cdots\hat{e_{k_i}}\cdots e_{k_m}e_{l_j}\otimes\big\langle e_{k_i},e_{l_1},\cdots,\hat{e_{l_j}}\cdots,e_{l_n}\big\rangle\\
&\quad+\sum_{i=1}^{m}(-1)^{1+1}\prod_{h=i+1}^{m}\varepsilon(|e_{k_i}|,|e_{k_h}|)\; e_{k_1}\cdots\hat{e_{k_i}}\cdots e_{k_m}e_{k_i}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle\\
&=\sum_{i,j=1}^{m,n}(-1)^j\prod_{h=i+1}^{m}\varepsilon(|e_{k_i}|,|e_{k_h}|)\,\varepsilon(|e_{k_i}|,|e_{l_j}|)\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\\
&\qquad e_{k_1}\cdots\hat{e_{k_i}}\cdots e_{k_m}e_{l_j}\otimes\big\langle e_{k_i},e_{l_1},\cdots,\hat{e_{l_j}}\cdots,e_{l_n}\big\rangle\\
&\quad+\sum_{i=1}^{m}\prod_{h=i+1}^{m}\varepsilon(|e_{k_i}|,|e_{k_h}|)\prod_{h=i+1}^{m}\varepsilon(|e_{k_h}|,|e_{k_i}|)\; e_{k_1}\cdots e_{k_i}\cdots e_{k_m}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle\\
&=\sum_{i,j=1}^{m,n}(-1)^j\prod_{h=i+1}^{m}\varepsilon(|e_{k_i}|,|e_{k_h}|)\,\varepsilon(|e_{k_i}|,|e_{l_j}|)\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\\
&\qquad e_{k_1}\cdots\hat{e_{k_i}}\cdots e_{k_m}e_{l_j}\otimes\big\langle e_{k_i},e_{l_1},\cdots,\hat{e_{l_j}}\cdots,e_{l_n}\big\rangle\\
&\quad+m\;e_{k_1}\cdots e_{k_i}\cdots e_{k_m}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle\,.
\end{aligned}$$

(The last equality uses the fact that $\prod_{h=i+1}^{m}\varepsilon(|e_{k_i}|,|e_{k_h}|)\;\prod_{h=i+1}^{m}\varepsilon(|e_{k_h}|,|e_{k_i}|)=1$.)

And we have

$$
\begin{aligned}
&t^p_{n-1}d^p_n(e_{k_1}\cdots e_{k_m}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle))\\
&=\sum_{i,j=1}^{m,n}(-1)^{j+1}\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\prod_{h=i+1}^{m-1}\varepsilon(|e_{k_i}|,|e_{k_h}|)\,\varepsilon(|e_{k_i}|,|e_{l_j}|)\\
&\qquad e_{k_1}\cdots\hat{e_{k_i}}\cdots e_{k_m}e_{l_j}\otimes\left\langle e_{k_i},e_{l_1},\cdots,\hat{e_{l_j}}\cdots,e_{l_n}\right\rangle\\
&\quad+\sum_{j=1}^{n}(-1)^{j+1}\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\,e_{k_1}\cdots e_{k_m}e_{l_j}\otimes\left\langle e_{l_1},\cdots,\hat{e_{l_j}}\cdots,e_{l_n}\right\rangle\\
&=-\sum_{i,j=1}^{m,n}(-1)^{j}\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\prod_{h=i+1}^{m-1}\varepsilon(|e_{k_i}|,|e_{k_h}|)\,\varepsilon(|e_{k_i}|,|e_{l_j}|)\\
&\qquad e_{k_1}\cdots\hat{e_{k_i}}\cdots e_{k_m}e_{l_j}\otimes\left\langle e_{k_i},e_{l_1},\cdots,\hat{e_{l_j}}\cdots,e_{l_n}\right\rangle\\
&\quad+\sum_{j=1}^{n}(-1)^{j+1}(-1)^{j+1}\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\prod_{h=1}^{j-1}\varepsilon(|e_{l_j}|,|e_{l_h}|)\,e_{k_1}\cdots e_{k_m}\otimes\left\langle e_{l_1},\cdots,e_{l_j}\cdots,e_{l_n}\right\rangle\\
&=-\sum_{i,j=1}^{m,n}(-1)^{j}\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\prod_{h=i+1}^{m-1}\varepsilon(|e_{k_i}|,|e_{k_h}|)\,\varepsilon(|e_{k_i}|,|e_{l_j}|)\\
&\qquad e_{k_1}\cdots\hat{e_{k_i}}\cdots e_{k_m}e_{l_j}\otimes\left\langle e_{k_i},e_{l_1},\cdots,\hat{e_{l_j}}\cdots,e_{l_n}\right\rangle\\
&\quad+n\,e_{k_1}\cdots e_{k_m}\otimes\left\langle e_{l_1},\cdots,e_{l_j}\cdots,e_{l_n}\right\rangle.
\end{aligned}
$$

(Here again we use $\prod_{h=1}^{j-1}\varepsilon(|e_{l_h}|,|e_{l_j}|)\ \prod_{h=1}^{j-1}\varepsilon(|e_{l_j}|,|e_{l_h}|)=1$.)

From that we deduce that

$$
(d^p_{n+1}t^p_n+t^p_{n-1}d^p_n)\ (e_{k_1}\cdots e_{k_m}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle)=(m+n)(e_{k_1}\cdots e_{k_m}\otimes\langle e_{l_1}\cdots e_{l_n}\rangle),\tag{4.11}
$$

We set

$$
t^p=\bigoplus_{n,m\in\mathbb{N},n+m=p\neq 0}\frac{1}{p}t^p_n,\tag{4.12}
$$

and thus, we deduce that

$$
d^pt^p+t^pd^p=\mathrm{Id}.\tag{4.13}
$$

Hence $\mathrm{H}_n(W^p)=0$ for all $p\geq 1$ and all $n$. □

The following theorem gives the color Koszul resolution.

**Theorem 3** *Then the sequence*

$$
C:\quad\cdots\to C_n\xrightarrow{d_n}C_{n-1}\to\cdots\to C_1\xrightarrow{\epsilon}C_0\to 0\tag{4.14}
$$

*is a $G$-graded $U(L)$-free resolution of the $G$-graded trivial module $\mathbb{K}$ via $\epsilon$.*

**Proof:** We consider the ($G$-graded) exact sequence of complexes

$$
0\to F_{p-1}\to F_pC\to W^p.\tag{4.15}
$$

For the associated long exact homology sequence it follows from Lemma 3 that

$$
\mathrm{H}_n(F_{p-1}C)\simeq\mathrm{H}_n(F_pC)\tag{4.16}
$$

for all $n$, and all $p\geq 1$. Since $F_0C$ is the graded complex $0\to\mathbb{K}\to\mathbb{K}\to 0$, we then obtain $\mathrm{H}_n(F_0C)=0$, for all $n$. Hence, by induction, $\mathrm{H}_n(F_pC)=0$ for all $n$ and all $p\geq 0$. Since $C=\cup_{p\geq 0}F_pC$ then the result follows that $\mathrm{H}_n(F_pC)=0$. □

## 4.2 Cohomology of color Lie Algebras

Let $M$ be a left $L$-module, we define the $n$-th graded cohomology group of $L$ with value in $M$ as

$$\mathrm{H}^n(L,M)_h := \mathrm{EXT}^n_{U(L)}(\mathbb{K},M)_h = \mathrm{Ext}^n_{U(L)\text{-gr}}(\mathbb{K},M[h]). \tag{4.17}$$

for all $h \in G$, where $\mathbb{K}$ is the trivial graded $L$-module, or equivalently, $U(L)$-module.

We also define

$$\mathrm{H}^n_{\mathrm{gr}}(L,M) = \mathrm{H}^n(L,M)_e. \tag{4.18}$$

Thus $\mathrm{H}^n(L,M)_h = \mathrm{H}^n_{\mathrm{gr}}(L,M[h])$.

Set

$$\mathrm{H}(L,M) = \oplus_{h\in G}\mathrm{H}^n(L,M)_h.$$

To compute $\mathrm{H}^n_{\mathrm{gr}}(L,M)$, we may be used the gr-free resolution of the trivial module $\mathbb{K}$ in Theorem 3. Let $M$ be a graded $L$-module, the cohomology groups $\mathrm{H}^n_{\mathrm{gr}}(L,M)$ are the homology groups of the complex

$$\begin{aligned}\mathrm{Hom}_{U(L)\text{-gr}}(C_n,M) &= \mathrm{Hom}_{U(L)\text{-gr}}(U(L)\otimes\wedge^n_\varepsilon L,M)\\ &\simeq \mathrm{Hom}_{\mathrm{gr}}(\wedge^n_\varepsilon L,M),\end{aligned}$$

where $C$ is the complex in Theorem 3. Under the above isomorphisms, the corresponding differential operator is given by

$$\delta^n(f)\,(x_1,\cdots,x_{n+1}) \tag{4.19}$$

$$= \sum_{i=1}^{n+1}(-1)^{i+1}\varepsilon_i\; x_i\cdot f\,(x_1,\cdots,\hat{x_i},\cdots,x_{n+1}) \tag{4.20}$$

$$+ \sum_{1\le i<j\le n+1}(-1)^{i+j}\varepsilon_i\varepsilon_j\varepsilon(|x_j|,|x_i|)\,f\,([x_i,x_j],x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{n+1})\,, \tag{4.21}$$

for all $f\in\mathrm{Hom}_{\mathrm{gr}}(\wedge^n_\varepsilon L,M)$, where the $\varepsilon_i$'s are given in 4.1. This description of the graded cohomology groups $\mathrm{H}^n_{\mathrm{gr}}(L,B)$ shows that these coincide, in the case of degree $e$, with the graded Cartan-Eilenberg cohomology of $L$ introduced by Scheunert and Zhang in([11],[13]).

Now we can apply Corollary 1 to the universal enveloping algebra $U(L)$ of a $G$-graded $\varepsilon$-Lie algebra $L$: by the Example in Section 3, we see $U(L)$ is a color Hopf algebra. Note that if $M$ is a graded $U(L)$-bimodule, the corresponding adjoint $L$-module ${}^{ad}(M)$ is given by (compare (3.74))

$$xm = x.m - \varepsilon(|x|,|m|)m.x \tag{4.22}$$

for homogeneous $x\in L$ and $m\in M$.

In summary, we get

**Theorem 4** *Let $L$ be a $G$-graded $\varepsilon$-Lie algebra, and let $U(L)$ be its universal enveloping algebra. Let $M$ be a graded $U(L)$-bimodule. Then there exists an isomorphism of graded spaces*

$$\mathrm{HH}^n(U(L),M)_h \simeq \mathrm{H}^n(L,{}^{ad}(M))_h = \mathrm{H}^n_{\mathrm{gr}}(L,{}^{ad}(M[h])).\quad n\ge 0. \tag{4.23}$$

*In particular we obtain*

$$\mathrm{HH}^n_{\mathrm{gr}}(U(L), M) \simeq \mathrm{H}^n_{\mathrm{gr}}(L, ^{ad}(M)), \quad n \geq 0. \tag{4.24}$$

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